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Full Research Article

Cost function and positive mathematical programming

Quirino Paris

Department of Agricultural and Resource Economics, University of California, Davis

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Abstract. A line of research in Positive Mathematical Programming (PMP) has pursued the goal of estimating a cost function capable of reproducing the base-year results in a sample of farms. Originally, the PMP approach estimated a "myopic" cost function, that is, a cost relation depending only on the output levels observed during a production cycle. No input price entered this type of cost function. In this paper we define and estimate a proper cost function that calibrates the economic results of a sample of farms. In the process, we demonstrate the existence of a unique solution of the PMP problem when observed output quantities and limiting input prices are taken as calibrating benchmarks. Furthermore, the paper shows how to obtain endogenous output supply elasticities that calibrate with available exogenous information in the form of previously estimated elasticities for an entire region or sector. This framework is applied to a sample of Italian farms that admit no production for some of the crop activities. This PMP model can be used to explore farmers' response to various policy decisions involving output prices, environmental constraints, limiting input supply, and other government interventions.

Keywords. Positive mathematical programming, solution uniqueness, supply elasticities, calibrating model

JEL codes. C6

1. Introduction

A cost function embodies the technological and market conditions facing a rational entrepreneur in the production of given output levels. Shephard lemma (1953) established the duality between a cost function and the underlying production technology. Under restrictive conditions, it may be possible to obtain an explicit expression of the underlying production function (Cobb-Douglas, CES, Generalized Leontief). In other cases, however, the derivation of an explicit production function may be impossible (Translog cost function). From an empirical and policy viewpoint, however, this lack of explicit duality is not a serious deficit since the cost function – as stated above – summarizes all the technological and market conditions.

Corresponding author: paris@primal.ucdavis.edu

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The methodological contribution of this paper – with reference to a sample of farms – can be outlined in four connected objectives. First, we assume the availability of information regarding a sample of farm production plans that were realized in the most current production cycles, as in the traditional approach of Positive Mathematical Programming (PMP). We also assume the availability of the price of limiting inputs (either at the farm or regional level). These important pieces of information are treated symmetrically and are used in a model that generates a unique calibrating solution. Second, we define and estimate a complete (non myopic) cost function that includes calibrating output quantities and limiting input prices. This cost function, then, can be used in the analysis of policy scenarios. Third, we calibrate the PMP farm model using exogenously determined (via econometric studies or expert judgement) output supply elasticities. Fourth, we extend the PMP methodology to the realistic case where not all farms cultivate all crop activities. The pursuit of these four objectives is the content of the following eight sections.

The paper proposes an approach to Positive Mathematical Programming that guarantees the uniqueness of the calibrating solution, a result that relies upon the use of all the available information, including prices of limiting inputs. This is the starting point of the paper. Toward the goal of dealing also with calibrating input prices we discuss first the PMP approach as often practiced to date (Qureshi et al., 2014, Arfini et al., 2013, Howitt et al., 2012, Henseler et al., 2009, Cortigiani et al., 2009).

The original formulation of the PMP methodology (Howitt, 1995a, 1995b) was based upon the estimation of the marginal cost associated with the observed production plan (or the difference between known per-output unit accounting costs and effective economic marginal costs). Phase I of this model took on the following specification (Howitt, 1995a, p. 151):

Primal	$\max TNR = \boldsymbol{p}'\boldsymbol{x} - \boldsymbol{c}'\boldsymbol{x}$		(1)
subject to	$A\boldsymbol{x} \leq \boldsymbol{b}$	structural constraints	(2)
	$\boldsymbol{x} \leq \overline{\boldsymbol{x}} + \varepsilon$	calibration constraints	(3)

and $\boldsymbol{x} \geq \mathbf{0}$ where A is a matrix of technical coefficients of dimensions $(I\times J,\ I<J)$ and all the other vectors are conformable to it. In particular, $\overline{\boldsymbol{x}} > \mathbf{0}$ is a vector of realized and observed levels of outputs whose utilization qualifies the positive feature of the PMP approach. Vector $\boldsymbol{b}$ refers to limiting input supplies. Vectors $\boldsymbol{p}$ and $\boldsymbol{c}$ represent market output prices and unit accounting costs, respectively. The parameter vector $\boldsymbol{\varepsilon}$ is composed of small, positive (user-determined) numbers whose role is to guarantee that the dual variables of the binding structural constraints achieve a positive value. In Howitt's words (1995a, p. 151): "The $\boldsymbol{\varepsilon}$ perturbation on the calibration constraints decouples the true resource constraints from the calibration constraints and ensures that the dual values on the allocable resources represent the marginal values of the resource constraints." This statement implies that, without the user-determined $\boldsymbol{\varepsilon}$ perturbation, the solution might result in the undesirable occurance of a zero dual variable for a binding resource constraint. Typically, the determination of the magnitude of the $\boldsymbol{\varepsilon}$ parameters requires a trial and error approach that is performed by solving repeatedly the phase I model until the user finds that the shadow (dual) prices of the binding resource constraints achieve positive values. With these stipulations, the dual of model (1)-(3) is stated as

Dual	$\min TC = \boldsymbol{b}'\boldsymbol{y} + \boldsymbol{\lambda}'[\overline{\boldsymbol{x}} + \boldsymbol{\varepsilon}]$	(4)
subject to	$A'\boldsymbol{y} + \boldsymbol{\lambda} + \boldsymbol{c} \geq \boldsymbol{p}$	(5)

with $\boldsymbol{y} \geq \mathbf{0}$, $\boldsymbol{\lambda} \geq \mathbf{0}$ where $\boldsymbol{y}$ represents the $(I\times1)$ vector of shadow prices of the structural constraints and the $(J\times1)$ vector $\boldsymbol{\lambda}$ represents the shadow prices of the calibration constraints. In the dual constraints (5) there are J constraints and $(I\times J)$ variables. At the optimal primal solution $\boldsymbol{x}^*$ relation (5) is satisfied with the equality sign by complementary slackness conditions given that $\boldsymbol{x}^* = \bar{\boldsymbol{x}} + \boldsymbol{\varepsilon} > \mathbf{0}$. Hence, the traditional specification of the PMP model is underdetermined (ill posed). It admits an infinite number of $(\boldsymbol{y}^*,\boldsymbol{\lambda}^*)$ solutions because there are more unknown variables than equations. This is the reason why the parameter $\boldsymbol{\varepsilon}$ was introduced in model (1)-(3) in order to elicit a dual solution with positive values of the shadow price $\boldsymbol{y}$ of the binding structural constraints. This means that I components of the vector $\boldsymbol{\lambda}$ assume a zero value.

Another criticism of the original PMP approach regards the specification of the calibration constraints. Why is the solution vector $\boldsymbol{x}$ of model (1)-(3) stated as less-than-or-equal to the observed vector of output levels $(\boldsymbol{x} \leq \bar{\boldsymbol{x}} + \boldsymbol{\varepsilon})$ in the calibration constraints (3)? The answer was (is): to guarantee a nonnegative dual vector of shadow prices $\boldsymbol{\lambda} \geq \mathbf{0}$ interpreted as variable marginal cost levels of $\bar{\boldsymbol{x}}$. Admittedly, this is an unsatisfactory answer. Given that vector $\bar{\boldsymbol{x}}$ represents observed (by the econometrician) output levels that were realized by the producer in a previous economic cycle, the observed vector $\bar{\boldsymbol{x}}$ may contain some deviations that either overstate or understate the levels of economically efficient production for a given farmer. A more plausible specification of the calibration constraints, therefore, could be $\boldsymbol{x} = \bar{\boldsymbol{x}} + \boldsymbol{h}$ where $\boldsymbol{h}$ is a conformable vector of unrestricted deviations from $\bar{\boldsymbol{x}}$.

Furthermore, a measure of the limiting input price vector $\bar{\boldsymbol{y}}$ may be available at either a regional or more local level. For example, the price of agricultural land is surely available, either by region or by area. The regional estimate may not be fitting every single farm but it can be assumed that it will fall within a reasonable range of the actual optimal land value of each farm as obtained by solving model (1)-(3). If the information on land price and other important limiting inputs is available, it should be used in a PMP approach in order to avoid violating the principal tenet of the methodology: all the available information should be used. Also in this case, therefore, it seems plausible to state a calibration constraint for the dual variable vector as $\boldsymbol{y} = \bar{\boldsymbol{y}} + \boldsymbol{u}$ where $\boldsymbol{u}$ is a conformable vector of unrestricted deviations from $\bar{\boldsymbol{y}}$.

Within this alternative PMP framework, the notion of a calibrating solution assumes a different structure from the original formulation of model (1)-(3). In that model, a calibrating solution achieves the obvious values of $\boldsymbol{x}^* = \bar{\boldsymbol{x}} + \boldsymbol{\varepsilon}$. Many critics of PMP have objected that this equation represents a tautology. In fact, the equality between the optimal solution of model (1)-(3) and the vector of observed output levels (adjusted by the $\boldsymbol{\varepsilon}$ parameter) is achieved because the – presumably – available information on the limiting input prices is ignored. With the more general specification of the calibration constraints in the form of $\boldsymbol{x} = \bar{\boldsymbol{x}} + \boldsymbol{h}$ and $\boldsymbol{y} = \bar{\boldsymbol{y}} + \boldsymbol{u}$, a calibrating solution $(\boldsymbol{x}^*,\boldsymbol{y}^*)$ will not, in general, be tautologically equal to $(\bar{\boldsymbol{x}},\bar{\boldsymbol{y}})$. But it can be arranged to make the solution $(\boldsymbol{x}^*,\boldsymbol{y}^*)$ as close as possible to the observed quantities and prices by using, for example, a least-squares minimization goal. This approach, then, resembles an econometric estimation where the goal is to minimize the residuals of a system of regressions. The objective of this alternative PMP methodology, therefore, is to make deviations $(\boldsymbol{h},\boldsymbol{u})$ as small as possible.

2. The use of $\bar{x}$ and $\bar{y}$ in PMP

To justify the structure of the novel phase I PMP model we begin with two preliminary analyses. First, suppose that a preliminary phase I of the PMP methodology is concerned with solving the following problem

$$\max TNR = \boldsymbol{p}'\boldsymbol{x} - \boldsymbol{c}'\boldsymbol{x} \tag{6}$$

$$\text{subject to} \quad A\boldsymbol{x} \le \boldsymbol{b} \quad \text{dual variables } \boldsymbol{y} \tag{7}$$

$$\boldsymbol{x} = \bar{\boldsymbol{x}} + \boldsymbol{h} \quad \text{dual variables } \boldsymbol{\lambda} \tag{8}$$

with $\boldsymbol{x} \ge \boldsymbol{0}$ and $\boldsymbol{h}$ unrestricted. Furthermore, we wish to minimize the sum of squared deviations, $\boldsymbol{h}'W\boldsymbol{h}/2$ as in a weighted least-squares approach. The W matrix is diagonal with elements $p_j > 0$ on the main diagonal, $j = 1,\dots,J$. The effective objective function, therefore, will be expressed as an auxiliary function such as $\max AUX = \boldsymbol{p}'\boldsymbol{x} - \boldsymbol{c}'\boldsymbol{x} - \boldsymbol{h}'W\boldsymbol{h}/2$. The purpose of the weight matrix W is to measure each component of the auxiliary objective function in the same measurement units, that is, in dollars. Forming the Lagrange function and stating the relevant Karush-Kuhn-Tucker (KKT) conditions will give

$$L = \boldsymbol{p}'\boldsymbol{x} - \boldsymbol{c}'\boldsymbol{x} - \boldsymbol{h}'W\boldsymbol{h}/2 + \boldsymbol{y}'[\boldsymbol{b} - A\boldsymbol{x}] + \boldsymbol{\lambda}'[\bar{\boldsymbol{x}} + \boldsymbol{h} - \boldsymbol{x}] \tag{9}$$

$$\frac{\partial L}{\partial \boldsymbol{x}} = \boldsymbol{p} - \boldsymbol{c} - A'\boldsymbol{y} - \boldsymbol{\lambda} \le \boldsymbol{0} \tag{10}$$

$$\frac{\partial L}{\partial \boldsymbol{h}} = -W\boldsymbol{h} + \boldsymbol{\lambda} = \boldsymbol{0} \tag{11}$$

From relation (11), $\boldsymbol{\lambda} = W\boldsymbol{h}$ and, thus, we can dispense from using the $\boldsymbol{\lambda}$ symbol explicitly. Relation (10), then, can be reformulated as

$$A'\boldsymbol{y} + W\boldsymbol{h} + \boldsymbol{c} \ge \boldsymbol{p}. \tag{12}$$

Relation (11) represents a case of self-duality, where a dual variable is equal (up to a scalar) to a primal variable.

Analogously, and still in a preliminary stage of analysis, let us consider the following problem

$$\min TC = \boldsymbol{b}'\boldsymbol{y} \tag{13}$$

$$\text{subject to} \quad A'\boldsymbol{y} + \boldsymbol{c} \ge \boldsymbol{p} \quad \text{dual variables } \boldsymbol{x} \tag{14}$$

$$\boldsymbol{y} = \bar{\boldsymbol{y}} + \boldsymbol{u} \quad \text{dual variables } \boldsymbol{\psi} \tag{15}$$

with $\boldsymbol{y} \ge \boldsymbol{0}$ and $\boldsymbol{u}$ as unrestricted deviations. Again, we wish to minimize the sum of squared deviations, $\boldsymbol{u}'V\boldsymbol{u}/2$ as in a weighted least-squares approach. The matrix V is diagonal with elements $b_i / \bar{y}_i > 0$ on the main diagonal, $i = 1,\dots,I$. The effective objective, then, will be expressed as an auxiliary function to be minimized such as $\min AUX2 = \boldsymbol{b}'\boldsymbol{y} + \boldsymbol{u}'V\boldsymbol{u}/2$. The purpose of the V matrix is to render homogeneous the measurement units of all the terms in the objective function and to scale the deviations $\boldsymbol{u}$ according to the size of the input constraints. Forming the Lagrange function and stating the relevant KKT conditions give

$$L^* = \boldsymbol{b}'\boldsymbol{y} + \boldsymbol{u}'V\boldsymbol{u}/2 + \boldsymbol{x}'[A'\boldsymbol{y} + \boldsymbol{c} - \boldsymbol{p}] + \boldsymbol{\psi}'[\boldsymbol{y} - \bar{\boldsymbol{y}} - \boldsymbol{u}] \tag{16}$$

$$\frac{\partial L^*}{\partial \boldsymbol{y}} = \boldsymbol{b} - A\boldsymbol{x} + \psi \ge \boldsymbol{0} \tag{17}$$

$$\frac{\partial L^*}{\partial \boldsymbol{u}} = V\boldsymbol{u} - \boldsymbol{\psi} = \boldsymbol{0} \tag{18}$$

From the self-dual relation (18), $\boldsymbol{\psi} = V\boldsymbol{u}$ and, again, we can dispense from using the symbol $\boldsymbol{\psi}$ explicitly. Thus, relation (17) can be reformulated as

$$A\boldsymbol{x} \leq \boldsymbol{b} + V\boldsymbol{u} \tag{19}$$

This discussion leads to a specification of a phase I PMP model that combines the duality relations of a LP problem together with the least-squares necessary conditions involving deviations $\boldsymbol{h}$ and $\boldsymbol{u}$. Combining constraints (12) and (19) with the calibration relations (8) and (15), we can write the relevant phase I PMP model as the problem of finding nonnegative vectors $\boldsymbol{x} \geq \boldsymbol{0}$, $\boldsymbol{y} \geq \boldsymbol{0}$ and unrestricted vectors $\boldsymbol{h}$ and $\boldsymbol{u}$ such that

$$A\boldsymbol{x} \leq \boldsymbol{b} + V\boldsymbol{u} \qquad \text{dual variables } \boldsymbol{y} \tag{20}$$
$$A'\boldsymbol{y} + W\boldsymbol{h} + \boldsymbol{c} \geq \boldsymbol{p} \qquad \text{dual variables } \boldsymbol{x} \tag{21}$$
$$\boldsymbol{x} = \overline{\boldsymbol{x}} + \boldsymbol{h} \qquad \text{dual variables } W\boldsymbol{h} \tag{22}$$
$$\boldsymbol{y} = \overline{\boldsymbol{y}} + \boldsymbol{u} \qquad \text{dual variables } V\boldsymbol{u} \tag{23}$$

together with the associated complementary slackness conditions. This PMP approach avoids the necessity of searching for the user-determined parameter $\boldsymbol{\varepsilon}$. The solution of model (20)-(23) generates estimates of the effective marginal cost **levels** $(A'\hat{\boldsymbol{y}}+W\boldsymbol{h}+\boldsymbol{c})$ and the input demand **levels** $A\hat{\boldsymbol{x}}$.

3. Solution uniqueness of the phase I PMP model

A least-squares (LS) solution is unique if and only if the matrix of "explanatory" variables has full rank. To verify this crucial condition in relation to model (20)-(23) we assume that vectors $\overline{\boldsymbol{x}}$ and $\overline{\boldsymbol{y}}$ have all positive components and thus $\boldsymbol{x} > \boldsymbol{0}$ and $\boldsymbol{y} > \boldsymbol{0}$ (this assumption will be relaxed in section 8). This implies – via complementary slackness conditions associated to relations (20)-(21) – that

$$A\boldsymbol{x} = \boldsymbol{b} + V\boldsymbol{u} \tag{24}$$
$$A'\boldsymbol{y} + \mathrm{W}\mathbf{h} + \mathbf{c} = \mathbf{p} \tag{25}$$

Substituting constraints (22) and (23) into (24) and (25), and rearranging terms in order to have all the unknowns on one side and the constant parameters on the other side of the equal sign, we obtain

$$-V\boldsymbol{u} + A\boldsymbol{h} = \boldsymbol{b} - A\overline{\boldsymbol{x}} \tag{26}$$
$$A'\boldsymbol{u} + W\boldsymbol{h} = \boldsymbol{p} - A'\overline{\boldsymbol{y}} - \boldsymbol{c} \tag{27}$$

and in matrix notation

$$\underset{M}{\begin{bmatrix} -V & A \\ A' & W \end{bmatrix}} \underset{\boldsymbol{z}}{\begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{h} \end{bmatrix}} \underset{=}{=} \underset{\boldsymbol{q}}{\begin{bmatrix} \boldsymbol{b} - A\overline{\boldsymbol{x}} \\ \boldsymbol{p} - A'\boldsymbol{y} - \boldsymbol{c} \end{bmatrix}} \tag{28}$$

The matrix M is of full rank because the nonsingular weight matrices V and W are on the main diagonal. Hence, the least-squares solution $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{h}}$ is unique. It follows that the solution $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ of model (20)-(23) is also unique. Given the structure of the M matrix, an inverse of M exists even if the A matrix is not of full rank.

The explicit least-squares solution of (28) is

$$\begin{bmatrix} \hat{\boldsymbol{u}} \\ \hat{\boldsymbol{h}} \end{bmatrix} = \begin{bmatrix} -(V+AW^{-1}A')^{-1} & V^{-1}A(A'V^{-1}A+W)^{-1} \\ W^{-1}A'(V+AW^{-1}A')^{-1} & (A'V^{-1}A+W)^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{b}-A\overline{\boldsymbol{x}} \\ \boldsymbol{p}-A'\overline{\boldsymbol{y}}-\boldsymbol{c} \end{bmatrix} \tag{29}$$

The optimal and calibrating LS levels of the primal and dual variables $\boldsymbol{x}$ and $\boldsymbol{y}$ then, are obtained as a simple addition according to the specification given in constraints (22) and (23) with $\hat{\boldsymbol{x}} = \overline{\boldsymbol{x}} + \hat{\boldsymbol{h}}$ and $\hat{\boldsymbol{y}} = \overline{\boldsymbol{y}} + \hat{\boldsymbol{u}}$.

4. Phase II: specification of a general cost function

Phase II of this PMP approach deals with the derivation of output marginal cost and input demand functions to be used in a calibrating model for the analysis of various policy scenarios. Following economic theory, we postulate that the total cost function of interest takes on the following symmetric and extended Leontief specification:

$$C(\boldsymbol{x},\boldsymbol{y}) = (\boldsymbol{g}'\boldsymbol{y})(\boldsymbol{f}'\boldsymbol{x}) + (\boldsymbol{g}'\boldsymbol{y})\boldsymbol{x}'Q\boldsymbol{x}/2 + (\boldsymbol{f}'\boldsymbol{x})[(\boldsymbol{y}^{1/2})'G\boldsymbol{y}^{1/2}] \tag{30}$$

where the $(J\times J)$ matrix Q is symmetric and positive definite, the $(I\times I)$ matrix G has elements $G_{i,i'} = G_{i',i} \geq 0$, $i\neq i'$. The elements $G_{i,i}$ can take on positive or negative values. The components of vectors $\boldsymbol{f}$ and $\boldsymbol{g}$ are free to take on any value. We require, however, that $\boldsymbol{f}'\boldsymbol{x} > 0$ and $\boldsymbol{g}'\boldsymbol{y} > 0$. From theory, a cost function is non-decreasing in output levels and input prices and, furthermore, it is homogeneous of degree one in input prices. This requirements drive to a large extent the specification of the cost function presented in relation (30). The vector of output marginal cost functions is stated as

$$MC_x = \frac{\partial C}{\partial \boldsymbol{x}} = (\boldsymbol{g}'\boldsymbol{y})\boldsymbol{f} + (\boldsymbol{g}'\boldsymbol{y})Q\boldsymbol{x} + \boldsymbol{f}[(\boldsymbol{y}^{1/2})'G\boldsymbol{y}^{1/2}] = A'\boldsymbol{y} + W\boldsymbol{h} + \boldsymbol{c} \tag{31}$$

while, by Shephard lemma, the vector of demand functions for inputs is stated as

$$\frac{\partial C}{\partial \boldsymbol{y}} = (\boldsymbol{f}'\boldsymbol{x})\boldsymbol{g} + \boldsymbol{g}(\boldsymbol{x}'Q\boldsymbol{x})/2 + (\boldsymbol{f}'\boldsymbol{x})\Delta(\boldsymbol{y}^{-1/2})'G\boldsymbol{y}^{1/2} = A\boldsymbol{x} \tag{32}$$

where the matrix $\Delta(\boldsymbol{y}^{-1/2})$ is diagonal with terms $y_i^{-1/2}$ on the main diagonal.

The vector of output supply functions comes from relation (31) by equating it to the vector of market output prices, $\boldsymbol{p}$ and inverting the marginal cost function to obtain

$$\boldsymbol{x} = -Q^{-1}\boldsymbol{f} - Q^{-1}\boldsymbol{f}[(\boldsymbol{y}^{1/2})'G\boldsymbol{y}^{1/2}]/(\boldsymbol{g}'\boldsymbol{y}) + Q^{-1}\boldsymbol{p}/(\boldsymbol{g}'\boldsymbol{y}) \tag{33}$$

that leads to the supply elasticity matrix

$$\mathrm{H} \equiv \Delta(\boldsymbol{p})\left[\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{p}}\right]\Delta(\boldsymbol{x}^{-1}) = \Delta(\boldsymbol{p})Q^{-1}\Delta(\boldsymbol{x}^{-1})/(\boldsymbol{g}'\boldsymbol{y}) \tag{34}$$

where matrices $\Delta(\boldsymbol{p})$ and $\Delta(\boldsymbol{x}^{-1})$ are diagonal with elements p_j and x_j^{-1}, respectively, on the main diagonals. Relation (34) includes all the own- and cross-price elasticities for all the output commodities admitted in the model.

The demand elasticities of limiting inputs can be easily measured from the input demand functions of relation (32). Suppose two limiting inputs form the structural constraints of the model. Then, the portion of the demand function that involves input prices can be stated as

$$b_1 + u_1 = K_1 + (\boldsymbol{f'x})[G_{11} + y_1^{-1/2}G_{12}y_2^{1/2}]$$
$$b_2 + u_2 = K_2 + (\boldsymbol{f'x})[G_{22} + y_1^{1/2}G_{12}y_2^{-1/2}] \tag{35}$$

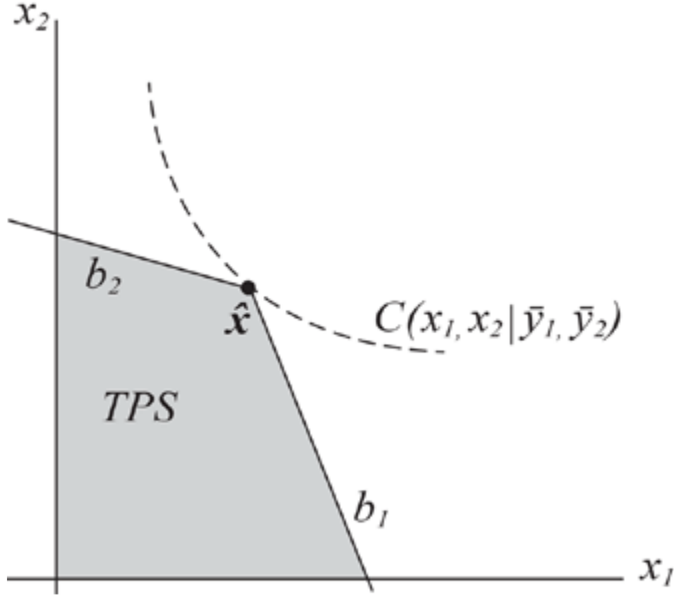
where K_1 and K_2 do not involve input prices. The matrix of derivatives of the demand functions results in

$$\begin{bmatrix} \frac{\partial(b_1+u_1)}{\partial y_1} & \frac{\partial(b_1+u_1)}{\partial y_2} \\ \frac{\partial(b_2+u_2)}{\partial y_1} & \frac{\partial(b_2+u_2)}{\partial y_2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}y_1^{-3/2}G_{12}y_2^{1/2} & \frac{1}{2}y_1^{-1/2}G_{12}y_2^{-1/2} \\ \frac{1}{2}y_1^{-1/2}G_{12}y_2^{-1/2} & -\frac{1}{2}y_1^{1/2}G_{12}y_2^{-3/2} \end{bmatrix}(\boldsymbol{f'x}) \tag{36}$$

This means that with only one limiting input, its demand elasticity will be equal to zero (as in a Leontief fixed coefficient specification) since the term G_{11} drops out of the derivative in (36).

An intuitive idea of how the estimated production plan $\hat{\boldsymbol{x}}$ lies on the cost function (30) is illustrated in Figure 1. For simplicity, given two outputs (x_1,x_2) and two inputs (b_1,b_2), Figure 1 shows the transformation possibility set, *TPS*, defined by two linear constraints involving the known levels of inputs b_1 and b_2. Assuming that the production plan $\hat{\boldsymbol{x}}$ maximizes the farm total revenue, it is possible to fit a cost function $C(x_1, x_2 | \bar{y}_1, \bar{y}_2)$ through the point $\hat{\boldsymbol{x}}$ where $\bar{y}_1, \bar{y}_2$ are given prices of inputs b_1 and b_2. The *TPS* is in general a convex set that is limited by known levels of inputs b_1 and b_2. Hence, the production plan $\hat{\boldsymbol{x}}$ must also be on the boundary of the true *TPS*(b_1,b_2) no matter what is the underlying technology corresponding to the cost function $C(x_1, x_2 | \bar{y}_1, \bar{y}_2)$ because the true *TPS*(b_1,b_2) is defined by the same known input levels b_1 and b_2 that define the *TPS* in Figure 1 and must go through the point $\hat{\boldsymbol{x}} = (\hat{x}_1, \hat{x}_2)$

Figure 1. Transformation possibility set and cost function.



5. Exogenous and disaggregated output supply elasticities

PMP has been applied frequently to analyze farmers' behavior to changes in agricultural policies. A typical empirical setting is to map out several areas, say T areas, in a region (or state) and to assemble a representative farm model for each area (or to treat each area as a large farm).

When supply elasticities are exogenously available (say the own-price elasticities of crops) at the regional (or state) level (via econometric estimation or other means), a connection of all area models with these exogenous elasticities can be specified by establishing a weighted sum of all the areas endogenous own-price elasticities and the given regional elasticities. The weights are the share of each area's revenue over the total revenue of the region.

Let us suppose that exogenous own-price elasticities of supply are available at the regional level for all the J crops, say $\bar{\eta}_j$, $j = 1,\ldots,J$. Then, the relation among these exogenous own-price elasticities and the corresponding areas' elasticities can be established as a weighted sum such as

$$\bar{\eta}_j = \sum_{t=1}^{T} w_{tj}\eta_{tj} \tag{37}$$

where the weights are the areas' revenue shares in the region (state)

$$w_{tj} = \frac{p_{tj}x_{tj}}{\sum_{s=1}^{T} p_{sj}x_{sj}} \tag{38}$$

and

$$\eta_{tj} = p_{tj}Q_t^{jj}x_t^{-1}/(\boldsymbol{g}'_t\boldsymbol{y}_t) \tag{39}$$

where Q_t^{jj} is the jth element on the main diagonal in the inverse of the Q_t matrix.

6. Estimation of the cost function parameters

Using the optimal LS solutions of $\boldsymbol{x},\boldsymbol{y},\boldsymbol{h}$ and $\boldsymbol{u}$ for each of the T areas, $\hat{\boldsymbol{x}}_t, \hat{\boldsymbol{y}}_t, \hat{\boldsymbol{h}}_t$ and $\hat{\boldsymbol{u}}_t$ obtained from solving phase I model (20)-(23), it is possible to proceed to the estimation of parameters $Q,G,\boldsymbol{f}$ and $\boldsymbol{g}$ of the cost function (30). The programming model that executes the estimation of the marginal cost (31) and input demand (32) functions in the presence of exogenous supply elasticities for a region (state) that is divided into T areas takes on the following least-squares specification:

$$\min LS = \sum_{t=1}^{T}(\boldsymbol{d}_t'\boldsymbol{d}_t + \boldsymbol{r}_t'\boldsymbol{r}_t)/2 \tag{40}$$

subject to

$(\boldsymbol{g}_t'\hat{\boldsymbol{y}}_t)\boldsymbol{f}_t + (\boldsymbol{g}_t'\hat{\boldsymbol{y}}_t)Q_t\hat{\boldsymbol{x}}_t + \boldsymbol{f}_t[(\hat{\boldsymbol{y}}_t^{1/2})'G_t\hat{\boldsymbol{y}}_t^{1/2}] + \boldsymbol{d}_t = A_t'\hat{\boldsymbol{y}}_t + W_t\hat{\boldsymbol{h}}_t + \boldsymbol{c}_t$ marginal cost function

$(\boldsymbol{f}_t'\hat{\boldsymbol{x}}_t)\boldsymbol{g}_t + \boldsymbol{g}_t(\hat{\boldsymbol{x}}_t'Q_t\hat{\boldsymbol{x}}_t)/2 + (\boldsymbol{f}_t'\hat{\boldsymbol{x}}_t)\Delta(\hat{\boldsymbol{y}}_t^{-1/2})'G_t\hat{\boldsymbol{y}}_t^{1/2} + \boldsymbol{r}_t = A_t\hat{\boldsymbol{x}}_t$ input demand function

$Q_t = L_tD_tL_t'$ positive semidefinitenes of Q_t

$Q_t\, Q_t^{-1} = L_t$ definiteness of Q_t

$\eta_{tjk} = \Delta(p_{tj})Q_t^{jk}\Delta(\hat{x}_{tk}^{-1})/(\boldsymbol{g}_t'\hat{\boldsymbol{y}}_t)$ endogenous own- and cross-supply elasticities

$$w_{tj} = \frac{p_{tj}\hat{x}_{tj}}{\sum_{s=1}^{T} p_{sj}\hat{x}_{sj}} \qquad \text{revenue shares}$$

$$\eta_{tj} = p_{tj} Q_t^{jj} \hat{x}_{tj}^{-1} / (\boldsymbol{g}_t' \hat{\boldsymbol{y}}_t) \qquad \text{endogenous own supply elasticities}$$

$$\bar{\eta}_j = \sum_{t=1}^{T} w_{tj}\eta_{tj} \qquad \text{disaggregation of exogenous elasticities}$$

with $D_t > 0$, $\boldsymbol{g}_t$ and $\boldsymbol{f}_t$ unrestricted parameters; $\boldsymbol{f}_t'\hat{\boldsymbol{x}}_t > 0$ and $\mathbf{g}_t'\hat{\mathbf{y}}_t > 0$, $\boldsymbol{d}_t \geq \mathbf{0}$, $\boldsymbol{r}_t \geq \mathbf{0}$. Vector variables $\boldsymbol{d}_t \geq \mathbf{0}$, $\boldsymbol{r}_t \geq \mathbf{0}$ perform the role of auxiliary slack variables that will equal to zero identically when minimized by the GAMS solver (the GAMS solver requires an explicit objective function). In this way, the system of relations involving the specification of marginal cost and demand functions for inputs will be estimated as they appear in equations (31) and (32). To limit the number of estimated parameters it may be convenient to assume that matrices Q and G belong to the entire area (or state) and do not carry the index t identifying each individual farm.

Model (40) is highly nonlinear in the constraints and a successful solution of it depends crucially on the proper scaling of the data series and on the choice of an initial point that falls in the neighborhood of the equilibrium solution. This specification was applied to three samples of T = 14 Italian farms (areas), classified according to acreage size, each producing four crops (sugar beet, soft wheat, corn and barley) using only land as a limiting input. The GAMS software program achieved an equilibrium solution in all the three cases.

In this paper we present the result for the class of farms of size greater than 100 hectares. Table 1 exhibits the observed output levels and the percent deviation obtained from solving model (20)-(23) (alternatively solving model (28)). The primal solution $\hat{\boldsymbol{x}}$ is almost equal to the observed output levels $\bar{\boldsymbol{x}}$ for every farm. All the percent deviations of the primal solution (except two) are well below the one percent level.

The same event characterizes the dual solution. Table 2 presents the deviations from the observed land input prices and the percent deviation of the optimal dual solution, $\hat{\boldsymbol{y}}$. Also in this case, the percent deviation is minimal in every farm.

The weighted LS minimization of the primal and dual deviations ($\boldsymbol{h}$,$\boldsymbol{u}$) has produced a largely satisfactory result in this sample. This goal is accomplished also by virtue of the diagonal weight matrices W and V.

The estimated parameters of the cost function are reported in Tables 3 and 4. For reasons of space, only three Q matrices are reported.

All 14 farms achieved a nonsingular $\hat{Q}$ matrix. This feature is instrumental in defining the matrix of endogenous supply elasticities. Table 5 presents the endogenous own- and cross-price supply elasticities for three farms.

We stipulated that regional, exogenous own-price supply elasticities were available in the magnitude of 0.5 for sugar beet, 0.4 for soft wheat, 0.6 for corn and 0.3 for barley. The endogenous own-price elasticities of all farms were aggregated to be consistent with the regional exogenous elasticities according to relation (37). Table 6 presents the farms' own-price supply elasticities and the revenue weights used in the aggregation relation.

Table 1. Observed output levels, $\bar{x}$ and percent deviation (dev) of the LS calibrated solution, $\hat{x}$.

Farm	Sugar Beet $\bar{x}_1$	Soft Wheat $\bar{x}_2$	Corn $\bar{x}_3$	Barley $\bar{x}_4$	Sugar Beet % dev	Soft Wheat % dev	Corn % dev	Barley % dev
1	1133.4240	305.4032	341.3693	18.2398	0.026	0.060	0.157	1.341
2	3103.7830	861.7445	478.4465	59.8025	0.016	0.042	0.052	0.637
3	1547.9780	450.7937	881.9748	7.6887	0.010	-0.003	0.011	0.164
4	3488.3540	821.3934	1493.332	51.1247	0.002	0.019	0.023	0.526
5	959.1102	468.2848	478.9261	28.2406	0.032	0.001	0.091	1.136
6	942.2039	801.1288	1283.591	152.581	0.049	0.059	0.046	0.384
7	1600.7310	695.8293	899.4739	66.9718	0.023	0.068	0.061	0.683
8	3507.5490	1212.8550	1237.584	98.0497	0.006	0.047	0.048	0.388
9	1050.5370	332.3773	498.0150	63.6696	0.043	0.188	0.120	0.846
10	3473.6780	952.5199	774.7402	84.0070	0.010	0.039	0.062	0.444
11	1245.7220	765.1689	501.9673	59.5366	0.030	0.047	0.101	0.718
12	3276.1450	1100.1680	742.9419	177.974	0.014	0.031	0.074	0.326
13	877.0970	380.9171	564.6091	76.2122	0.048	0.055	0.105	0.683
14	1430.9460	768.6901	1309.392	67.7906	0.026	0.038	0.035	0.604

Table 2. Deviations of $\hat{y}$ from $\bar{y}$: vector $\hat{u}$.

Farm	Absolute Deviation $\hat{u}$	Observed Land Prices $\bar{y}$	Percent Deviation %
1	0.0053817	4.42	0.122
2	0.0026860	4.38	0.061
3	0.0004449	6.98	0.006
4	0.0018006	5.73	0.031
5	0.0031117	4.40	0.071
6	0.0014600	1.86	0.078
7	0.0032416	3.65	0.089
8	0.0018922	3.36	0.056
9	0.0052767	2.75	0.192
10	0.0027213	4.28	0.064
11	0.0029836	3.28	0.091
12	0.0011904	1.93	0.062
13	0.0028811	2.32	0.124
14	0.0022795	4.03	0.057

7. Calibrating equilibrium model

With the estimates of the cost function parameters $\hat{f}, \hat{g}, \hat{Q}, \hat{G}$ it is possible to formulate a calibrating equilibrium model for each farm (sector, area) of the following structure

$$\min CSC_t = z'_{pt} y_t + z'_{dt} x_t = 0 \tag{41}$$

Table 3. Intercepts $\hat{f}$, $\hat{g}$ and $\hat{G}$ matrix of the marginal cost and input demand functions.

Farm	$\hat{f}$				$\hat{g}$	$\hat{G}$	$\hat{f}'\hat{y}$	$\hat{g}'\hat{y}$
	Sugar Beet	Soft Wheat	Corn	Barley				
1	0.1110	0.0912	-0.0727	0.6183	0.00191	-1.2669	140.294	0.00847
2	-0.0112	0.6636	-0.0784	0.7116	0.00110	-0.9190	542.721	0.00484
3	0.5937	0.6745	0.4603	1.0948	0.00222	-0.0313	1637.550	0.01550
4	-0.0549	0.2153	0.2302	1.0018	0.00061	-1.0016	380.591	0.00350
5	0.0174	0.5424	-0.0297	0.6213	0.00361	-0.5182	274.242	0.01589
6	-0.7008	1.6163	6.0416	1.0965	0.01073	-0.2639	8561.452	0.01998
7	0.2155	0.2852	0.2414	0.9854	0.00328	-0.7473	827.362	0.01198
8	-0.0769	0.7406	0.4971	0.9387	0.00791	-0.9633	1336.688	0.02660
9	-0.0559	0.7323	0.3863	0.8988	0.00941	-1.2263	435.449	0.02592
10	0.0300	0.8861	-0.2342	0.9465	0.00055	-0.6090	846.879	0.00234
11	0.2427	0.4817	-0.0555	0.9430	0.00638	-0.7449	699.825	0.02095
12	0.0796	0.8584	0.4385	1.0428	0.00650	-1.4001	1717.901	0.01255
13	0.7831	0.3158	-0.3314	0.8222	0.00711	-0.9164	683.351	0.01651
14	0.1802	0.6635	0.1982	0.9977	0.00925	-1.2669	1095.888	0.03732

Table 4. Matrices $\hat{Q}$ and $\hat{D}$ for three farms.

	Matrix $\hat{Q}$				Matrix $\hat{D}$			
	Sugar Beet	Soft Wheat	Corn	Barley	Sugar Beet	Soft Wheat	Corn	Barley
Farm 1								
S. Beet	0.90363	-1.97461	-0.88227	0.06447	0.90363			
S.Wheat	-1.97461	5.83223	2.23097	0.35591		1.51732		
Corn	-0.88227	2.23097	1.49261	0.17779			0.57068	
Barley	0.06447	0.35591	0.17779	21.75286				21.55051
Farm 2								
S. Beet	0.71517	-2.04607	-0.76495	-0.00159	0.71517			
S.Wheat	-2.04607	7.25493	2.41519	-0.05663		1.40123		
Corn	-0.76495	2.41519	1.53268	-0.03759			0.67780	
Barley	-0.00159	-0.05663	-0.03759	18.98344				18.97949
Farm 3								
S. Beet	1.24597	0.24597	-2.27223	-0.42147	1.24597			
S.Wheat	0.24597	1.95471	-1.25809	-0.02225		1.90615		
Corn	-2.27223	-1.25809	4.76858	0.85444			0.28099	
Barley	-0.42147	-0.02225	0.85444	6.78018				6.59126

Table 5. Endogenous own- and cross-supply elasticities for three farms.

	Sugar Beet	Soft Wheat	Corn	Barley
Farm 1				
S. Beet	0.2001	0.1952	0.1321	-0.1091
S. Wheat	0.2815	0.6056	-0.2563	-0.1763
Corn	0.2052	-0.2760	1.2485	-0.1514
Barley	-0.0089	-0.0100	-0.0079	0.5927
Farm 2				
S. Beet	0.2487	0.2182	0.1855	0.0133
S. Wheat	0.3081	0.4399	-0.2513	0.0162
Corn	0.1454	-0.1395	1.5539	0.0191
Barley	0.0012	0.0011	0.0023	0.4196
Farm 3				
S. Beet	0.1839	0.1379	0.1727	-0.1676
S. Wheat	0.2347	0.3725	0.2474	-0.5665
Corn	0.4893	0.4121	0.4952	-0.9548
Barley	-0.0044	-0.0087	-0.0088	2.5417

Table 6. Disaggregation/aggregation of the regional, endogenous supply elasticities.

Farms	Endogenous Own-Supply Elasticities				Revenue Weights			
	Sugar Beet: 0.5	Soft Wheat: 0.4	Corn: 0.6	Barley: 0.3	Sugar Beet	Soft Wheat	Corn	Barley
1	0.2001	0.6056	1.2485	0.5927	0.0406	0.0291	0.0295	0.0165
2	0.2487	0.4399	1.5539	0.4196	0.1334	0.0937	0.0489	0.0628
3	0.1839	0.3725	0.4952	2.5417	0.0527	0.0446	0.0699	0.0070
4	0.2225	0.4774	0.5665	0.9868	0.1000	0.0893	0.1383	0.0536
5	0.1599	0.4512	0.8430	0.5691	0.0326	0.0413	0.0385	0.0256
6	0.6932	0.9332	0.5011	0.1080	0.0371	0.0828	0.1151	0.1601
7	0.0990	0.2906	0.3522	0.1918	0.0502	0.0688	0.0769	0.0606
8	0.1347	0.2714	0.2307	0.0823	0.1288	0.1292	0.1022	0.0931
9	0.1303	0.2670	0.3384	0.1544	0.0376	0.0335	0.0426	0.0576
10	0.2954	0.3940	1.9745	0.9603	0.1027	0.0930	0.0649	0.0825
11	0.1085	0.3682	0.2755	0.2195	0.0424	0.0737	0.0417	0.0539
12	0.1843	0.2692	0.2407	0.1197	0.1555	0.1079	0.0685	0.1868
13	0.0947	0.2861	0.4050	0.1486	0.0299	0.0336	0.0454	0.0689
14	0.0883	0.2455	0.3772	0.1349	0.0564	0.0795	0.1175	0.0711

subject to
$$(\hat{\boldsymbol{f}}_t'\boldsymbol{x}_t)\hat{\boldsymbol{g}}_t + \hat{\boldsymbol{g}}_t(\boldsymbol{x}_t'\hat{Q}_t\boldsymbol{x}_t)/2 + (\hat{\boldsymbol{f}}_t'\boldsymbol{x}_t)\Delta(\boldsymbol{y}_t^{-1/2})'\hat{G}_t\boldsymbol{y}_t^{1/2} + \boldsymbol{z}_{pt} = \boldsymbol{b}_t + V_t\hat{\boldsymbol{u}}_t$$
$$(\boldsymbol{g}_t'\boldsymbol{y}_t)\hat{\boldsymbol{f}}_t + (\boldsymbol{g}_t'\boldsymbol{y}_t)\hat{Q}_t\boldsymbol{x}_t + \hat{\boldsymbol{f}}_t[(\boldsymbol{y}_t^{1/2})'\hat{G}_t\boldsymbol{y}_t^{1/2}] = \boldsymbol{p}_t + \boldsymbol{z}_{dt}$$
with $\boldsymbol{x}_t \geq \boldsymbol{0}$, $\boldsymbol{y}_t \geq \boldsymbol{0}$, $\boldsymbol{z}_{pt} \geq \boldsymbol{0}$, $\boldsymbol{z}_{dt} \geq \boldsymbol{0}$. The variables $\boldsymbol{z}_{pt}$ and $\boldsymbol{z}_{dt}$ are slack-surplus variables of the primal and dual constraints, respectively. The objective function (*CSC*) of model (41) combines all the complementary slackness conditions of the farm (region, area). Hence, its optimal value must be equal to zero. The solution of the equilibrium model (41) produces optimal values of the primal and dual variables, $\boldsymbol{x}_t$ and $\boldsymbol{y}_t$ that are identical to the solution values of model (20)-(23). Notice that the matrix of constant technical coefficients, A_t, no longer appears in the calibrating equilibrium model (41). This elimination removes the last vestige of a linear structure that has been considered too rigid for representing the choices of a producer. Model (41) can be used to perform response analysis to variations in prices, subsidies, quotas, input quantities, and other parameters for a variety of policy scenarios.

8. PMP uniqueness with missing observations

Empirical reality compels a further consideration of the above methodology in order to deal with farm samples where not all farms produce all commodities. It turns out that very little must be changed for obtaining a unique and calibrating solution in the presence of missing commodities, their prices and the corresponding technical coefficients.

To exemplify, suppose that the farm sample displays the following Table 7 of observed crop levels.

Table 7. Observed Output Levels, $\bar{\boldsymbol{x}}$ with non produced commodities.

Farm	Sugar Beet	Soft Wheat	Corn	Barley
	$\bar{x}_1$	$\bar{x}_2$	$\bar{x}_3$	$\bar{x}_4$
1	1133.4240	0.0	341.3693	18.2398
2	3103.7830	861.7445	0.0	59.8025
3	0.0	450.7937	881.9748	0.0
4	3488.3540	821.3934	1493.332	51.1247
5	959.1102	468.2848	0.0	28.2406
6	942.2039	801.1288	1283.591	152.581
7	1600.7310	0.0	899.4739	66.9718
8	0.0	1212.8550	1237.584	98.0497
9	1050.5370	332.3773	0.0	63.6696
10	3473.6780	952.5199	774.7402	0.0
11	0.0	765.1689	501.9673	59.5366
12	3276.1450	1100.1680	0.0	177.974
13	877.0970	380.9171	564.6091	76.2122
14	1430.9460	0.0	1309.392	0.0

Other missing information deals with prices and unit accounting costs associated with the zero-levels of crops. Furthermore, the technical coefficients of the farms not producing the observed crops also equal to zero. Hence, we can state that, for $t = 1,\ldots,T$, the number of farms, and $j = 1,\ldots,J$, the number of crops, if $\bar{x}_{tj} = 0$ also $p_{tj} = 0$, $c_{tj} = 0$ and $A_{tij} = 0$. Furthermore, suppose that only one input, land, is involved in this farm sample. Then, the land price is observed for all farms.

As to the solution of the Phase I PMP specification, we expect that $x_{tj} = \bar{x}_{tj} + h_{tj}$ for $\bar{x}_{tj} > 0$ and $h_{tj} = x_{tj} = 0$ for $\bar{x}_{tj} = 0$. It turns out that the least-squares computation of the deviations u_{ti} and h_{tj} expressed by equation (29) produces the desired estimates of the deviations h_{tj} and crop levels x_{tj} when the observed level of those crops equals zero, $\bar{x}_{tj} = 0$. This is so because the first term on the RHS of (29) is equal to zero by construction, $b_i - \sum_{j=1}^{J} A_{ij}\bar{x}_j = b_i - \sum_{j=1}^{J} (acres_{ij} / \bar{x}_j)\bar{x}_j = 0$. The second term on the RHS of (29) reduces to zero because of the zero information about non-produced crops, $p_j - \sum_{i=1}^{I} A_{ij}\bar{y}_i - c_j = 0 - 0\bar{y}_i - 0 = 0$ Therefore, $\hat{h}_{tj} = \hat{x}_{tj} = 0$ for $\bar{x}_{tj} = 0$ and the least-squares PMP solution is unique also in this more elaborate case.

The estimation of the cost function carries through as in section 6 without modification. Also the phase III calibrating model expressed in (41) needs no adjustment.

9. Results for a farm sample with missing production of some crops

The observed crop production of a 14-farm sample is given in Table 7. Also the corresponding output prices, $p_{tj} = 0$ and accounting costs, $c_{tj} = 0$ are part of the data sample for the no-production levels $\bar{x}_{tj} = 0$ as reported in Table 7. Furthermore, $A_{tij} = 0$ for the same activities of no-production.

Table 8 presents the unique least-squares estimates of the crop levels and the corresponding percentage deviation from the observed sample data.

Table 8. Estimated Output Levels, $\bar{x}$ and Percent Deviation (dev) for the sample with missing crop production (compare with Table 7).

Farm	Sugar Beet $\bar{x}_1$	Soft Wheat $\bar{x}_2$	Corn $\bar{x}_3$	Barley $\bar{x}_4$	Sugar Beet % dev	Soft Wheat % dev	Corn % dev	Barley % dev
1	1133.7140	0	341.9053	18.4843	0.0256	0	0.1570	1.3400
2	3104.2820	862.1098	0.0000	60.1834	0.0161	0.0424	0	0.6369
3	0	450.7820	882.0680	0	0	-0.0026	0.0106	0
4	3488.4150	821.5529	1493.6830	51.3938	0.0017	0.0194	0.0235	0.5264
5	959.4208	468.2891	0	28.5614	0.0324	0.0009	0	1.1360
6	942.6667	801.6001	1284.1790	153.1671	0.0491	0.0588	0.0458	0.3840
7	1601.1000	0	900.0223	67.4290	0.0231	0	0.0610	0.6825
8	0	1213.4210	1238.1750	98.4298	0	0.0466	0.0478	0.3876
9	1050.9910	333.0022	0	64.2084	0.0433	0.1880	0	0.8463
10	3474.0410	952.8955	775.2208	0	0.0105	0.0394	0.0620	0
11	0	765.5305	502.4727	59.9640	0	0.0473	0.1007	0.7179
12	3276.6110	1100.5140	0	178.5547	0.0142	0.0314	0	0.3260
13	877.5201	381.1268	565.2019	76.7330	0.0482	0.0550	0.1050	0.6833
14	1431.3200	0	1309.8500	0	0.0261	0	0.0350	0

Except for two cells, the percent deviations of the estimated crop levels from the observed production quantities are below 1 percent. The cells with a zero estimated quantity level correspond to the cells with observed zero level of production, as in Table 7. Table 9 presents the estimated land price and the percent deviation from the observed input price.

Table 9. Deviations of $\hat{y}$ from $\bar{y}$.

Farm	Estimated Land Prices $\hat{y}$	Observed Land Prices $\bar{y}$	Percent Deviation %
1	4.428035	4.42	0.1818
2	4.382827	4.38	0.0645
3	6.980315	6.98	0.0045
4	5.731801	5.73	0.0314
5	4.402587	4.40	0.0588
6	1.861460	1.86	0.0785
7	3.653809	3.65	0.1044
8	3.362198	3.36	0.0654
9	2.756308	2.75	0.2294
10	4.281756	4.28	0.0410
11	3.283229	3.28	0.0984
12	1.931129	1.93	0.0585
13	2.322881	2.32	0.1242
14	4.031362	4.03	0.0338

The deviations of the estimated land prices from the observed prices are all below one percent. Table 10 presents the estimates of the parameters of the cost function under the condition of zero production for some crops in various farms.

Table 11 presents the own price elasticities of the 14 farms that correspond to the observed and exogenous price elasticities of the four crops.

The calibrating model (41) applies also to this data sample without any modification.

10. Conclusion

We have achieved the objective of using all the available information about output quantities and limiting input prices, and the formulation of a calibrating PMP model that is free of the rigidities of a linear programming structure. In the process, we dispense with the necessity of dealing with the user-determined vector of small and arbitrary positive numbers $\boldsymbol{\varepsilon}$ that is required by the traditional PMP methodology. We also demonstrate the uniqueness of the calibrating solution. Two empirical examples were presented. In the first sample of 14 farms and 4 crops, all farms produce every commodity. In the second sample, some of the farms do not produce all the commodities. This is the typical case. It is shown that the uniqueness of the calibrating solution is maintained also in this more elaborate case.

Table 10. Intercepts $\hat{f}$, $\hat{g}$ and $\hat{G}$ matrix of the marginal cost and input demand functions for the case of zero production of some crop in various farms.

Farm	$\hat{f}$				$\hat{g}$	$\hat{G}$	$\hat{f}\hat{y}$	$\hat{g}'\hat{y}$
	Sugar Beet	Soft Wheat	Corn	Barley				
1	-0.15426	0.00274	0.73961	-0.11144	0.00686	-1.9508	75.930	0.03038
2	0.07435	-0.14574	0.03714	0.32875	0.00495	-3.1350	124.945	0.02171
3	0.03532	0.27086	-0.11507	0.03964	0.00052	-1.0871	20.595	0.00363
4	-0.02920	0.07372	0.10372	0.85513	0.00441	-2.4222	157.570	0.02530
5	0.02132	0.01858	0.11481	0.06645	0.00754	-2.3602	31.051	0.03318
6	0.22974	0.22787	-0.02587	0.26590	0.01186	-5.3411	406.732	0.02208
7	0.13824	-0.00074	-0.14506	0.29086	0.00273	-3.6725	110.382	0.00997
8	0.01525	0.40319	-0.12078	-0.17109	0.01373	-3.0222	322.849	0.04616
9	0.11620	-0.08339	0.00722	-0.03553	0.00499	-2.9037	92.071	0.01375
10	-0.00636	0.36252	-0.14708	0.00406	0.00076	-2.4324	209.320	0.00325
11	0.00236	0.30162	-0.12984	-0.24600	0.00764	-2.8158	150.906	0.02507
12	0.10700	0.16788	0.00002	0.57873	0.00004	-2.6811	638.676	0.00001
13	0.24139	0.42193	-0.25400	-0.19708	0.01046	-2.9149	213.946	0.02431
14	0.05358	0.06649	0.07745	0.01622	0.00686	-2.6086	178.140	0.03038

Table 11. Disaggregation/aggregation of the regional, endogenous supply elasticities when some crops are not produced in various farms.

Farms	Endogenous Own-Supply Elasticities				Revenue Weights			
	Sugar Beet: 0.5	Soft Wheat: 0.4	Corn: 0.6	Barley: 0.3	Sugar Beet	Soft Wheat	Corn	Barley
1	0.257	0	0.385	0.722	0.0523	0	0.0368	0.0198
2	0.289	0.409	0	0.251	0.1719	0.1139	0	0.0748
3	0	0.515	1.740	0	0	0.0542	0.0871	0
4	1.873	0.262	0.428	0.333	0.1288	0.1086	0.1726	0.0639
5	0.381	0.577	0	0.445	0.0421	0.0502	0	0.0306
6	0.052	0.221	0.322	0.120	0.0479	0.1007	0.1437	0.1904
7	0.149	0	0.656	0.225	0.0647	0	0.0960	0.0723
8	0	0.329	0.294	0.122	0	0.1571	0.1275	0.1108
9	0.212	0.407	0	0.335	0.0485	0.0408	0	0.0688
10	0.241	0.309	0.794	0	0.1323	0.1130	0.0810	0
11	0	0.399	0.313	0.268	0	0.0896	0.0520	0.0643
12	0.487	0.714	0	0.531	0.2004	0.1311	0	0.2220
13	0.142	0.325	0.837	0.259	0.0385	0.0409	0.0567	0.0822
14	0.305	0	0.583	0	0.0727	0	0.1465	0

The central piece of the methodology is the estimation of a non-myopic cost function defined over output levels and input (shadow) prices. This cost function is not associated with an explicit functional form of the underlying technology. For this reason, the phase I procedure estimates the output and limiting input shadow price **levels** that are consistent with a linear technology and the observed information about output levels and input prices. These **levels**, then, are used to estimate the parameters of the cost **function**.

This model is akin to an econometric model that is estimated for prediction without regards to the identification of the structural parameters of the cost function. The model "goodness," then, depends on the ability to predict outside the sample observations. This test can be executed with multiple observations per farm.

When several observations per each sample farm are available, the estimation procedure becomes a proper econometric approach. In this case, it will be convenient to split the sample observations in two parts: say, ninety percent (or whatever share of the observations the researcher would prefer) for estimating the cost function and ten percent for evaluating the prediction ability of the PMP methodology. This approach is a goal of further research.

The PMP procedure presented in this paper uses also exogenous information about supply elasticities assumed to be available at a regional or state level. It shows how to calibrate the endogenous elasticities to this additional information while achieving a unique calibrating solution.

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